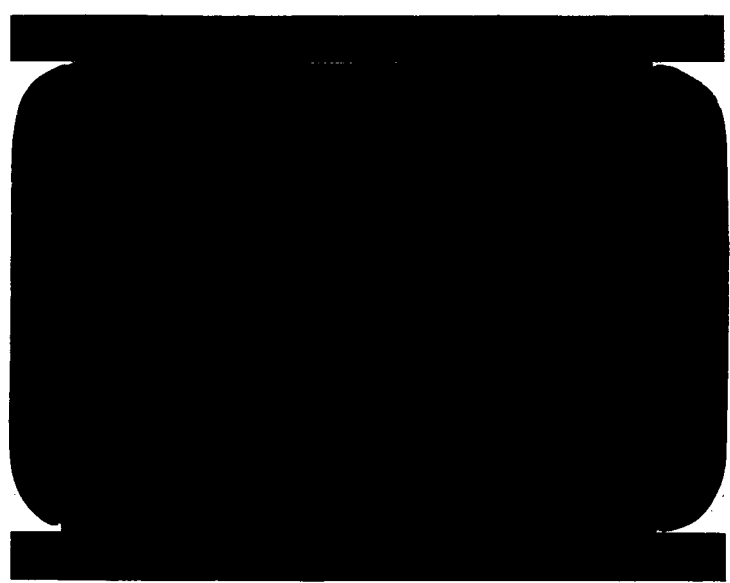


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**FREQUENCY ANALYSIS OF STIFFENED
CYLINDRICAL SHELLS WITH
SIMPLY-SUPPORTED EDGES**

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FOREWORD

This report presents an analysis of stiffened circular cylinders. The analysis results in a solution for the natural frequencies of a simply-supported cylindrical shell that is reinforced with frames and stringers. The solution is used to calculate natural frequencies for the F-1 and AC-2 Centaur adapters.

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SYMBOLS

a, a_f	= mean radius of skin, frames
a_{ij}	= elements of the frequency determinant; $i, j = 1, 2, 3$
A^f, A^s	= cross-sectional area of frames, stringers
A, B, C	= maximum amplitude of the displacements u, v, w
D_s	= determinant of symmetric part of a_{ij}
D_Δ	= difference between determinant a_{ij} and D_s
e_x^s, e_s^f	= distance of frame, stringer centroid from skin middle surface
E	= Young's modulus
f	= superscript used to identify frame quantities
h	= skin thickness
h_s^{**}, h_f^{**}	= defined by equations (19a,b)
h_s^*, h_f^*	= defined by equations (23a,b)
I^s, I^f	= stringer, frame moment of inertia
K_m^x, K_m^s	= $1 + (h_s^{**}/h)^3, 1 + (h_f^{**}/h)^3$
k_n^x, k_n^s	= $1 + h_s^*/h, 1 + h_f^*/h$
k	= $(\rho'/E)(1-\nu^2)$
L	= length of cylinder
M_x, M_s	= bending stress couples
M_{xs}, M_{sx}	= twisting stress couples
M_{xs}^*	= $1/2(M_{xs} - M_{sx})$
m	= axial mode number, number of half waves in the axial direction
M	= $m\pi/L$
N_x, N_s, N_{xs}, N_{sx}	= stress resultants
n	= circumferential mode number, number of full waves in the circumferential direction
N	= n/a
Q	= any one of the shell parameters such as h, I^f , etc.
Q_x, Q_s	= transverse shearing stress resultants

q_x, q_s, q_z	= externally applied forces (including inertia forces) in the x, s and z directions
s	= superscript used to identify stringer quantities, also arc length
t	= time
u, v, w	= displacements of the middle surface of the skin in the x, s, z directions
w_s, w_t	= structural, total weight of the stiffened cylinder
x, s, z	= coordinates in the longitudinal, circumferential, and radially inward directions as shown in Figure 1.
α	= $1/12 (h/a)^2$
Δ^s, Δ^f	= distance between stringers, frames
Δ_1, Δ_2	= $a_{13} - a_{31}, a_{23} - a_{32}$
ν	= Poisson's ratio
ξ	= stretching component of the natural frequency
ρ	= mass density of the skin
ρ'	= (mass of stiffened cylinder)/(2 $\pi a L h$)
ϕ	= derived quantity, see equation (36)
ω	= circular frequency
ω_r	= reference frequency
$\underline{\omega}, \underline{n}, \underline{N}$	= circular frequency and circumferential mode numbers at the minimum circular frequency (for a given value of axial mode number, m)
$a^2 k \omega^2$	= reduced circular frequency

SUMMARY

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The stress resultant equations of motion and moment equilibrium for a cylindrical shell are modified to allow for axial and bending rigidity of frames and stringers. Appropriate stress resultant-displacement equations are formulated thus allowing the transformation of stress resultant equations to displacement equations to be made. The assumption of a simply-supported solution yields the frequency equation, which is used to calculate frequencies for the F-1 and AC-2 Centaur adapters, proposed versions of the AC-4,5 Centaur adapters, and OAO aft fairing. Since the natural frequencies of the F-1 adapter have been determined experimentally by the Fort Worth test on a full size adapter it was possible to check the predictive value of the analytic frequency equation. Correlation between calculated frequencies and test frequencies of the F-1 adapter is good, thus lending confidence to the value of the analysis.

Appendix A is included to display the analysis of the behavior of the derived frequency equation. This leads to an approximate equation which relates, in a simple manner, lowest natural frequency and shell parameters.

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1.0 INTRODUCTION

The accurate determination of the circular frequencies of vibration of a complex structure is usually a time consuming and tedious task. Usually one is forced to pursue an analysis that breaks the structure into small elements. This results in a matrix relation between the forces and deflections of the elements that must be inverted to obtain a frequency equation. Except for the simplest of structures, this inversion and the evaluation of the resulting frequency determinant requires the use of a computer. The initial breakdown of the structure is done by hand and is consequently very time consuming for typical structures. The use of a computer for the inversion of the matrix and calculation of the frequency determinant certainly speeds the calculations, but care must be exercised to limit the size of the matrix so that the computer can successfully perform the inversion. In short, although this method usually provides accurate results, the process requires much hand calculation, large amounts of computer time, and is therefore costly and contains many opportunities for error.

In an attempt to determine frequencies more quickly, the results of published papers are oftentimes adapted to the problem at hand. The problem with this approach is that the existing analyses are primarily for very simple structures. The effects of, for example, stiffeners are difficult to determine and the results obtained by modifying the results for less complex structures are often grossly in error.

The foregoing remarks are intended to elucidate the need for a method which will provide a means of calculating, accurately and quickly, the natural frequencies to be used in predesign analyses. The need for such a method provided the impetus for the investigation that resulted in the analysis presented in this report. Although it is assumed that the structure is somewhat uniform the effect of stiffeners is accounted for in a logical manner. The frequency analysis of stiffened circular cylindrical shells follows.

2.0 EQUATIONS THAT CHARACTERIZE THE ELASTO-KINETIC DEFORMATION OF STIFFENED CYLINDRICAL SHELLS

The stress resultant equations of motion that characterize the deformation of the elastic circular cylinders will be recorded from GD/A Report #ERR-AN-137 (Reference 1). These equations will be modified to apply to stiffened cylinders and appropriate stress resultant-displacement relations will be obtained, thus allowing the equations of motion to be written in terms of the displacements.

The equations of motion for cylindrical shells are

$$\frac{\partial N_x}{\partial x} - \frac{\partial N_{sx}}{\partial s} + q_x = 0 \quad (1a)$$

$$\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} - \frac{Q_s}{a} + q_s = 0 \quad (1b)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_s}{\partial s} + \frac{N_s}{a} + q_z = 0 \quad (1c)$$

$$\frac{\partial M_{xs}}{\partial s} - \frac{\partial M_s}{\partial x} + Q_s = 0 \quad (2a)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{sx}}{\partial s} - Q_x = 0 \quad (2b)$$

$$N_{xs} + N_{sx} + M_{sx} / a = 0 \quad (2c)$$

where the spacial coordinates x, s, z are defined by Figure 1 and the stress resultants ($N_x, N_s, N_{xs}, N_{sx}, Q_x, Q_s$) and stress couples (M_x, M_s, M_{ks}, M_{sx}) are defined by Figure 2. For the analysis of the free (no forcing function, no body forces) vibrations of circular cylinders q_x, q_s, q_z become the inertia forces

$$q_x = -\rho h \frac{\partial^2 u}{\partial t^2} \quad (3a)$$

$$q_s = -\rho h \frac{\partial^2 v}{\partial t^2} \quad (3b)$$

$$q_z = -\rho h \frac{\partial^2 w}{\partial t^2} \quad (3c)$$

where ρ is the mass density of the shell material, h is the thickness of the shell, t is time, and u, v, w are the displacements in the x, s, z - coordinate directions. In the analysis that follows the longitudinal inertia terms (3a,b) are set equal to zero. This is a simplification based on the results of a shallow shell analysis by E. Reissner (Reference 2). Utilizing Reissner's simplification identifies this analysis as an investigation of the breathing modes of vibration.

Consider a cylinder that is reinforced in the circumferential and longitudinal directions. It is assumed that the stiffeners (frames and stringers) are spaced close enough together so that the equations of motion (1,2) may be applied to an element of discrete dimensions. Furthermore, it is assumed that the element abcd (Figure 1) is typical and may be used to represent any element of the shell. Allowing these assumptions, the forces and moments resulting from the rigidity of the stiffeners may be included in the equations of motion.

The stress resultants in the force equations of motion (1a,b,c) due to frame and stringer deformation enter the equations of motion in the same way as the skin stress resultants. Therefore, we may write directly equations (1a,b,c) as they apply to stiffened cylindrical shells. Equations (1a,b,c) become

$$\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_s^s}{\partial s} \right) - \frac{\partial N_{sx}}{\partial s} = 0 \quad (4a)$$

$$\frac{\partial N_{xs}}{\partial x} + \left(\frac{\partial N_s}{\partial s} + \frac{\partial N_s^f}{\partial s} \right) - \frac{1}{a} (Q_s + Q_s^f) = 0 \quad (4b)$$

$$\left(\frac{\partial Q_s}{\partial s} + \frac{\partial Q_s^f}{\partial s} \right) + \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_x^s}{\partial x} \right) + \frac{1}{a} (N_s + N_s^f) + q_z = 0 \quad (4c)$$

where the superscripts (s, f) on the stress resultants and stress couples identify the stress resultants and stress couples due to the deformation of the stringers and frames, respectively. The stress resultants and stress couples due to the deformation of the skin carry no superscripts.

The derivation of the equations of moment equilibrium analagous to (2a, 2b) requires more care because of the additional terms introduced by the moments of the stiffener stress resultants about the center line of the skin. The stress resultants due to stiffener deformation are assumed to act at the stiffener's centroid (line no, Figures 3,4). Since the equations of moment equilibrium (2a,b) are obtained by summing moments about the middle line of the skin, the moments of the stiffener stress resultants about the middle line of the skin must be considered. Although the frequency calculations for the Centaur adapters show that these terms are small in comparison to terms from the basic equations (2a,b) there was no apriori reason for neglecting them in the analysis. Figure 3 shows the stress resultants (but not stress couples) due to frame deformation. Summation of moments about point p yields

$$M^f = \left(2Q_s^f + \frac{\partial Q_s^f}{\partial s} \Delta s \right) \Delta x a_f \text{TAN} \frac{\Delta s}{2a} - \left(\frac{\partial N_s^f}{\partial s} \Delta s \Delta x \right) \left\{ e_f + h/2 - a_f \left[\left(\cos \frac{\Delta s}{2a} \right)^{-1} - 1 \right] \right\} \quad (5)$$

When the usual small angle approximations

$$\text{TAN} \frac{\Delta s}{2a} \approx \frac{\Delta s}{2a} \quad (6)$$

$$\cos \frac{\Delta s}{2a} \approx 1 - \frac{1}{2} \left(\frac{\Delta s}{2a} \right)^2 \quad (7)$$

are made and it is assumed that

$$\frac{\frac{\partial Q_s^f}{\partial s} \Delta s}{Q_s^f} \ll 1 \quad (8)$$

equation (5) becomes

$$M^f = \left(Q_s^f - \frac{\partial N_s^f}{\partial s} \bar{e}_f \right) \Delta s \Delta x \quad (9)$$

where

$$\bar{e}_f = e_f + h/2 - (\Delta s)^2/8a \quad (10)$$

and e_f is the distance from the centroid of the frame to the surface of the skin that is in contact with the frame.

In a similar manner the moments due to the stringer stress resultants (Figure 4) are summed, yielding

$$M^s = - \left(Q_x^s + \frac{\partial N_x^s}{\partial x} \bar{e}_s \right) \Delta x \Delta s \quad (11)$$

where

$$\bar{e}_s = e_s + h/2 \quad (12)$$

and e_s is the distance from the centroid of the stringer to the surface of the skin that is in contact with the stringer. The expressions for M^s and M^f and the stress couples M_x^s , M_s^f are added to the equations of moment equilibrium (2a,b). With these additions, (2a,b) become

$$\frac{\partial M_{xs}}{\partial x} - \left(\frac{\partial M_s}{\partial x} + \frac{\partial M_s^f}{\partial x} \right) + \left(Q_s + Q_s^f \right) - \frac{\partial N_s^f}{\partial s} \bar{e}_f = 0 \quad (13a)$$

$$\frac{\partial M_{sx}}{\partial s} + \left(\frac{\partial M_x}{\partial s} + \frac{\partial M_x^s}{\partial s} \right) - \left(Q_x + Q_x^s \right) - \frac{\partial N_x^s}{\partial x} \bar{e}_s = 0 \quad (13b)$$

Equation (2c) is unchanged.

The shear stress resultants $(Q_x + Q_x^s, Q_s + Q_s^f)$ are eliminated by substituting equations (13a,b) into (4b,c). With this substitution the pertinent equations of motion become

$$\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_s}{\partial x} \right) - \frac{\partial N_{sx}}{\partial s} = 0 \quad (14a)$$

$$\frac{\partial N_{xs}}{\partial x} + \frac{1}{a} \frac{\partial M_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} + \frac{\partial N_s^f}{\partial s} \left[1 - \frac{\bar{\epsilon}_f}{a} \right] - \frac{1}{a} \left(\frac{\partial M_s}{\partial s} + \frac{\partial M_s^f}{\partial s} \right) = 0 \quad (14b)$$

$$\begin{aligned} & \left(\frac{N_s}{a} + \frac{N_s^f}{a} \right) + \left(\frac{\partial^2 M_{sx}}{\partial x \partial s} - \frac{\partial^2 M_{xs}}{\partial x \partial s} \right) + \left(\frac{\partial^2 M_s}{\partial s^2} + \frac{\partial^2 M_s^f}{\partial s^2} \right) + \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_x^f}{\partial x^2} \right) \\ & + \left(\frac{\partial^2 N_s^f}{\partial s^2} - \frac{\partial^2 N_s}{\partial x^2} \right) + q_z = 0 \end{aligned} \quad (14c)$$

Since $\bar{\epsilon}_f/a$ will be much less than one for all applications, the bracketed term in (14b) will be taken as one.

The displacement equations of motion are obtained by substituting the appropriate stress resultant-displacement relations into equations (14a,b,c). The stress resultant strain relations are taken from the aforementioned GD/A Report #ERR-AN-137. Essentially, these relations are the classic equations of thin shell theory that are based on the small deflection analysis of isotropic elastic thin shells whose displacements satisfy Kinkhoff's hypotheses and whose stresses normal to the middle surface are small. These relations are

$$N_x = \frac{Eh}{(1-\nu^2)} \left[\frac{\partial u}{\partial x} + \nu \left(\frac{\partial v}{\partial s} - \frac{w}{a} \right) \right] \quad (15a)$$

$$N_s = \frac{Eh}{(1-\nu^2)} \left[\nu \frac{\partial u}{\partial x} + \left(\frac{\partial v}{\partial s} - \frac{w}{a} \right) \right] \quad (15b)$$

$$N_{xs} + M_{sx}/a = -N_{sx} = + \frac{Eh}{2(1+\nu)} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} \right) \quad (15c)$$

$$M_x = \frac{-Eh^3}{12(1-\nu^2)} \left[\frac{\partial^2 w}{\partial x^2} + \nu \left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{a} \frac{\partial v}{\partial s} \right) \right] \quad (16a)$$

$$M_s = \frac{-Eh^3}{12(1-\nu^2)} \left[\nu \frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{a} \frac{\partial v}{\partial s} \right) \right] \quad (16b)$$

$$M_{xs}^* = \frac{1}{2} (M_{xs} - M_{sx}) = + \frac{Eh^3}{12(1+\nu)} \left(\frac{\partial^2 w}{\partial x \partial s} + \frac{1}{a} \frac{\partial v}{\partial x} \right) \quad (16c)$$

where u , v , w are the displacements of the middle surface of the skin, h is the skin thickness, ν is Poisson's ratio and E is Young's modulus.

In accord with Mushtari's simplification we reject the displacements u and v in the expressions for M_x , M_s , M_{xs} . With this simplification, equations (16a,b,c) become

$$M_x = - \frac{Eh^3}{12(1-\nu^2)} \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial s^2} \right] \quad (17a)$$

$$M_s = - \frac{Eh^3}{12(1-\nu^2)} \left[\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial s^2} \right] \quad (17b)$$

$$M_{xs}^* = + \frac{Eh^3}{12(1+\nu)} \frac{\partial^2 w}{\partial x \partial s} \quad (17c)$$

From Hooke's law and elementary beam theory we deduce the stress resultants and stress couples in the stiffeners to be

$$N_x^s = EA^s (\Delta^s)^{-1} \epsilon_x^s \quad (18a)$$

$$N_s^f = EA^f (\Delta^f)^{-1} \epsilon_s^f \quad (18b)$$

$$M_x^s = -EI^s (\Delta^s)^{-1} \frac{\partial^2 w^s}{\partial x^2} \quad (18c)$$

$$M_s^f = -EI^f \left(1 + \frac{\bar{e}_f}{a} \right)^2 (\Delta^f)^{-1} \frac{\partial^2 w^f}{\partial s^2} \approx -EI^f (\Delta^f)^{-1} \frac{\partial^2 w^f}{\partial s^2} \quad (18d)$$

where Young's modulus (E) is assumed to be the same for skin and stiffeners. A^s and A^f , I^s and I^f , w^s and w^f , Δ^s , Δ^f , are, respectively, the cross-sectional area, moment of inertia, radial displacement of the centroid, and spacing of stringers and frames. If it is assumed that the radial deflection (w^s , w^f) of the stiffeners is equal to the deflection, w , of the skin equations (18c,d) become

$$M_x^s = -EI^s(\Delta^s)^{-1} \frac{\partial^2 w}{\partial x^2} = - \frac{Eh_s^{3**}}{12(1-\nu^2)} \frac{\partial^2 w}{\partial x^2} \quad (19a)$$

$$M_s^f = -EI^f(\Delta^f)^{-1} \frac{\partial^2 w}{\partial x^2} = - \frac{Eh_f^{3**}}{12(1-\nu^2)} \frac{\partial^2 w}{\partial s^2} \quad (19b)$$

where h_f^{3**} and h_s^{3**} are introduced to simplify the expressions for $M_x + M_x^s$ and $M_s + M_s^f$. It follows from (17a,b) and (19a,b) that

$$M_x + M_x^s = - \frac{Eh^3}{12(1-\nu^2)} \left[\nu \frac{\partial^2 w}{\partial s^2} + K_m^x \frac{\partial^2 w}{\partial x^2} \right] \quad (20a)$$

$$M_s + M_s^f = - \frac{Eh^3}{12(1-\nu^2)} \left[\nu \frac{\partial^2 w}{\partial x^2} + K_m^s \frac{\partial^2 w}{\partial s^2} \right] \quad (20b)$$

where

$$K_m^x = 1 + h_s^{3**}/h^3 \quad (21a)$$

$$K_m^s = 1 + h_f^{3**}/h^3 \quad (21b)$$

If in addition it is assumed that

$$u = u^s \quad (22a)$$

$$v = v^f \quad (22b)$$

the stiffener stress resultants become

$$N_x^s = EA^s(\Delta^s)^{-1} \frac{\partial u}{\partial x} = \frac{Eh_s^*}{(1-\nu^2)} \frac{\partial u}{\partial x} \quad (23a)$$

$$N_s^f = EA^f(\Delta^f)^{-1} \left(\frac{\partial v}{\partial s} - \frac{w}{a} \right) = \frac{Eh_f^*}{(1-\nu^2)} \left(\frac{\partial v}{\partial s} - \frac{w}{a} \right) \quad (23b)$$

and the expressions for $(N_x + N_x^s)$ and $(N_s + N_s^f)$ become

$$(N_x + N_x^s) = \frac{Eh}{(1-\nu^2)} \left[K_n^x \frac{\partial u}{\partial x} + \nu \left(\frac{\partial v}{\partial s} - \frac{w}{a} \right) \right] \quad (24a)$$

$$(N_s + N_s^f) = \frac{Eh}{(1-\nu^2)} \left[\nu \frac{\partial u}{\partial x} + K_n^s \left(\frac{\partial v}{\partial s} - \frac{w}{a} \right) \right] \quad (24b)$$

where

$$K_n^x = 1 + h_s^* / h \quad (25a)$$

$$K_n^s = 1 + h_f^* / h \quad (25b)$$

The equations of motion may now be written in terms of the displacements u, v, w .

3.0 SIMPLY-SUPPORTED SOLUTION

The displacements

$$u = A \cos(MX) \cos(NS) e^{i\omega t} \quad (26a)$$

$$v = B \sin(MX) \sin(NS) e^{i\omega t} \quad (26b)$$

$$w = C \sin(MX) \cos(NS) e^{i\omega t} \quad (26c)$$

where A, B, C are constants, $M = m\pi/L$, $N = n/a$, m and n integers, L and a the length and mean radius of the cylinder, ω the circular frequency satisfy the simply-supported conditions

$$v(x=0) = v(x=L) = w(x=0) = w(x=L) = 0 \quad (27)$$

Substituting the displacements (26a,b,c) into the displacement equations of motion results in a set of three equations which may be represented symbolically as

$$a_{11}A + a_{12}B + a_{13}C = 0 \quad (28a)$$

$$a_{21}A + a_{22}B + a_{23}C = 0 \quad (28b)$$

$$a_{31}A + a_{32}B + (a_{33} - k\omega^2)C = 0 \quad (28c)$$

where

$$k = (\rho'/E)(1-\nu^2) \quad (29a)$$

$$\rho' = (\text{mass of stiffened cylinder})/(2\pi a L h) \quad (29b)$$

and the elements, a_{ij} , of equations (28a,b,c) are

$$a_{11} = M^2 K_n^x + N^2(1-\nu)/2 \quad (30a)$$

$$a_{12} = -MN(1+\nu)/2 \quad (30b)$$

$$a_{13} = -M\nu/a \quad (30c)$$

$$a_{21} = -MN(1+\nu)/2 \quad (30d)$$

$$a_{22} = M^2(1-\nu)/2 + N^2 K_n^s \quad (30e)$$

$$a_{23} = NK_n^s(a)^{-1} + M^2 N^2 a \alpha(2-\nu) + N^3 a \alpha K_m^s \approx K_n^s N(a)^{-1} \quad (30f)$$

$$a_{31} = -M\nu(a)^{-1} - M^3 \epsilon_s \frac{h_s^*}{h} \quad (30g)$$

$$a_{32} = NK_n^s(a)^{-1} - N^3 \epsilon_f \frac{h_f^*}{h} \quad (30h)$$

$$a_{33} = \alpha a^2 \left[M^4 K_m^x + N^4 K_m^s + 2M^2 N^2 + K_n^s (\alpha a^4)^{-1} \right] - N^2 \epsilon_f(a)^{-1} \frac{h_f^*}{h} \quad (30j)$$

The equations (28a,b,c) have non-trivial solutions for A, B, C only if

$$\text{determinant } a_{ij} = 0 \quad (31)$$

Writing (31) as a near-symmetric determinant, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} - \Delta_1 & a_{23} - \Delta_2 & a_{33} - k_m^2 \end{vmatrix} = 0 \quad (32)$$

where

$$\Delta_1 = M^3 \epsilon_s \frac{h_s^*}{h} \quad (33a)$$

$$\Delta_2 = N^3 \epsilon_f \frac{h_f^*}{h} \quad (33b)$$

Expanding (32) yields

$$k_m^2 = D_s + D_\Delta \quad (34a)$$

where

$$D_s = a_{33} + \frac{2a_{23}a_{12}a_{13} - (a_{11}a_{23}^2 + a_{22}a_{13}^2)}{a_{11}a_{22} - a_{12}^2} \quad (34b)$$

and

$$D_\Delta = \frac{\Delta_2(a_{11}a_{23} - a_{12}a_{13}) - \Delta_1(a_{12}a_{23} - a_{22}a_{13})}{a_{11}a_{22} - a_{12}^2} \quad (34c)$$

The frequency equation in its desired form follows from equations (34a,b,c) and definitions (30a-30j). i.e.,

$$a^2 k^2 = + \frac{K_n^s K_n^x - v^2}{K_n^x + \phi} + \alpha a^4 (M^4 K_m^x + N^4 K_m^s + 2M^2 N^2) \\ + \frac{\frac{h_f}{h} \cdot a N^2 \epsilon_f \left(v \frac{N^2}{M^2} - K_n^x \right) + a M^2 \epsilon_s \left(-v + \frac{N^2}{M^2} K_n^s \right) \frac{h_s}{h}}{K_n^x + \phi} \quad (35)$$

where

$$\phi = \frac{2N^2}{M^2} \left(\frac{K_n^x K_n^s - v}{1-v} \right) + \frac{N^4}{M^4} K_n^s \quad (36)$$

4.0 APPLICATION OF FREQUENCY EQUATION TO CENTAUR INTERSTAGE ADAPTER

The physical properties of the two adapter configurations (Table I) were used in equation (35) to calculate natural frequencies. Since the structural and total weight (structural weight + component weight) of the AC-2 Centaur adapter differ by such a large amount (1117 lbs/1539 lbs), the calculated frequencies for this adapter are presented for both weights (Table III). The structural weight of the F-1 Centaur adapter (570 lbs) is close enough to the weight of the adapter tested at Fort Worth (623 lbs) to allow one calculation, based on the average weight, to suffice. The results of this calculation are presented in Table II. The results of the Fort Worth test on a full size adapter are included for comparison.

5.0 DISCUSSION

It is the author's opinion that this analysis is a logical approach to the problem of stiffened cylinders. The high degree of correlation between the frequencies predicted by equation (35) and those determined experimentally at Fort Worth indicate the ability of equation (35) to predict vibration frequencies of reinforced cylinders. For this reason and the fact that the frequency equation (35) provides a quick, slide-rule determination of the frequencies, the results of this analysis should prove to be a useful tool for predicting the natural frequencies of vibration of stiffened cylinders.

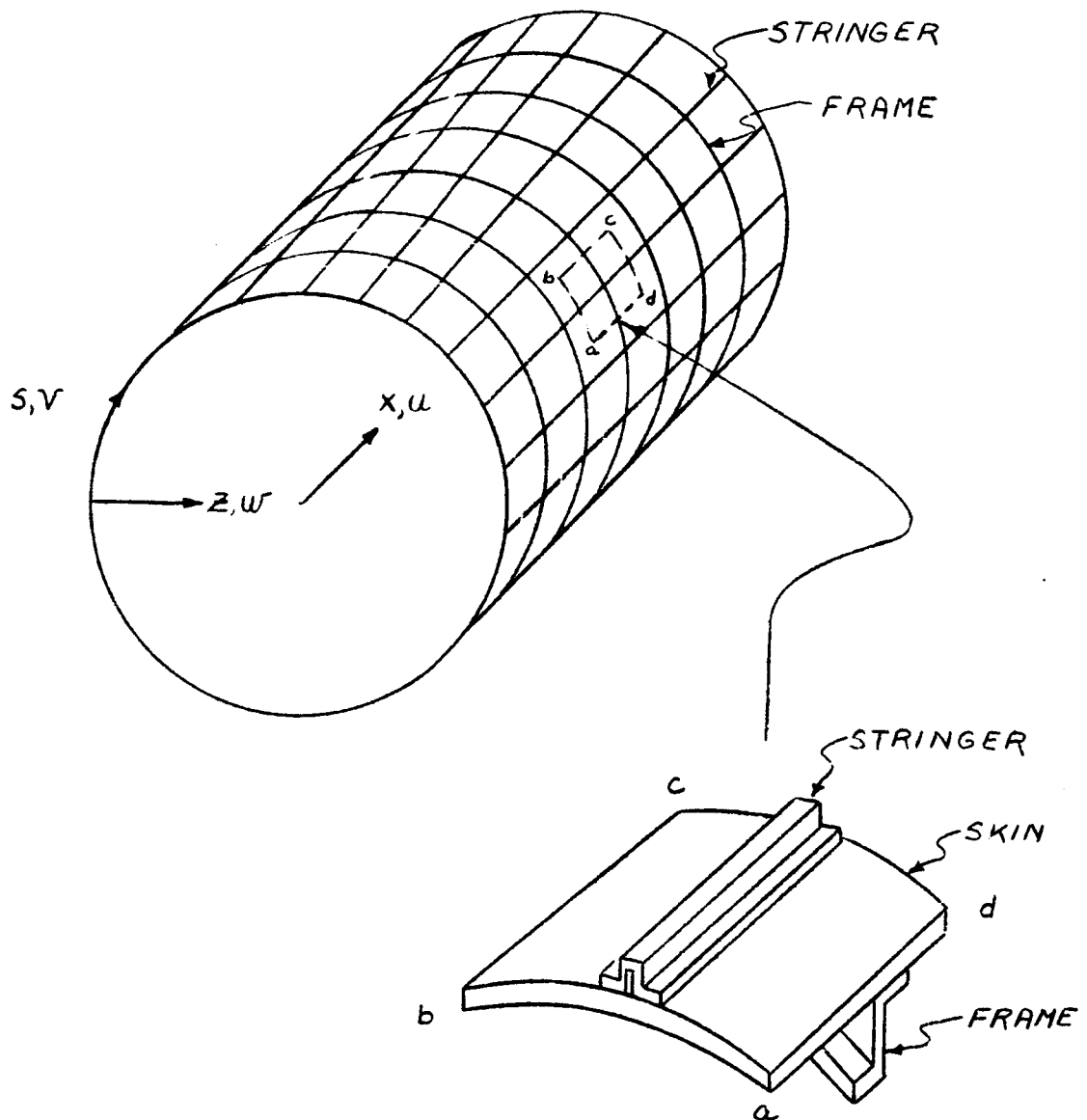


FIGURE 1. COORDINATE SYSTEM AND GENERIC ELEMENT
FOR A STIFFENED CIRCULAR CYLINDER

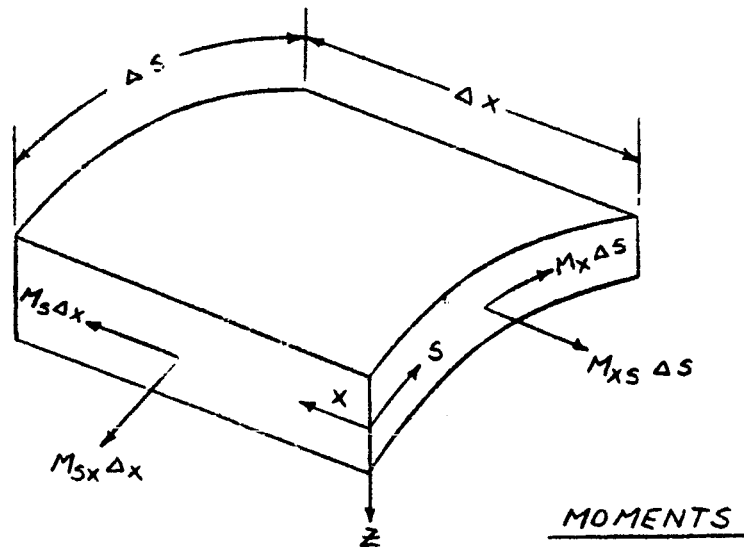
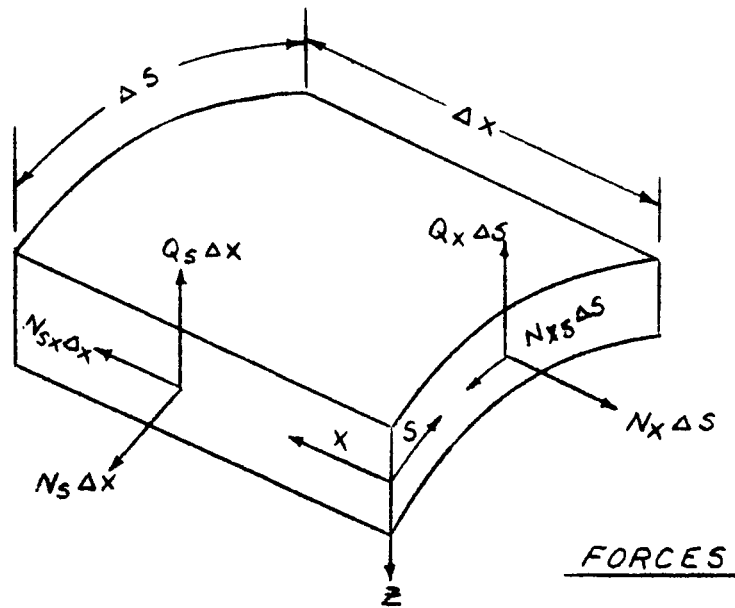


FIGURE 2. STRESS RESULTANTS AND STRESS COUPLES

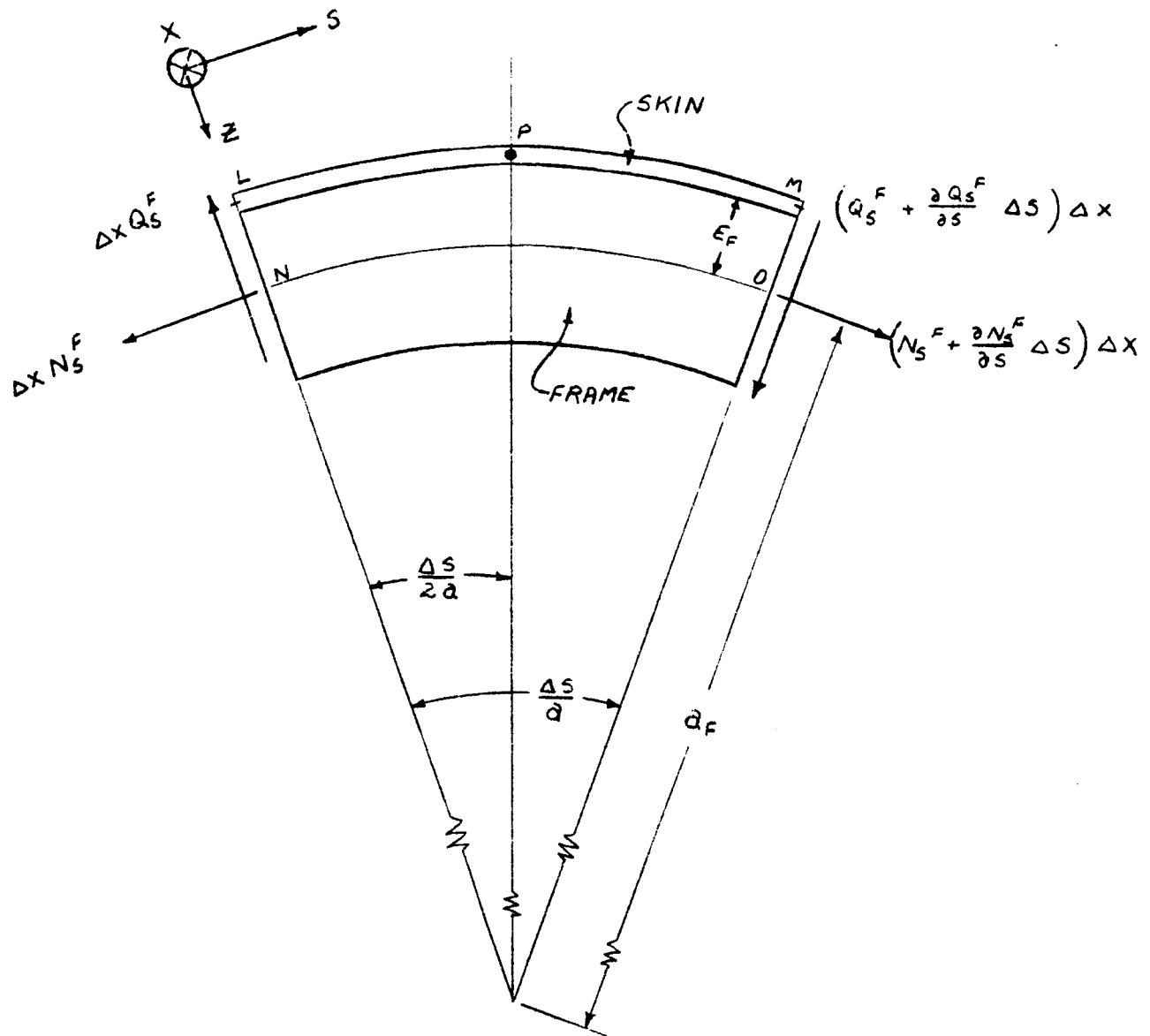


FIGURE 3. STRESS RESULTANTS DUE TO FRAME DEFORMATION

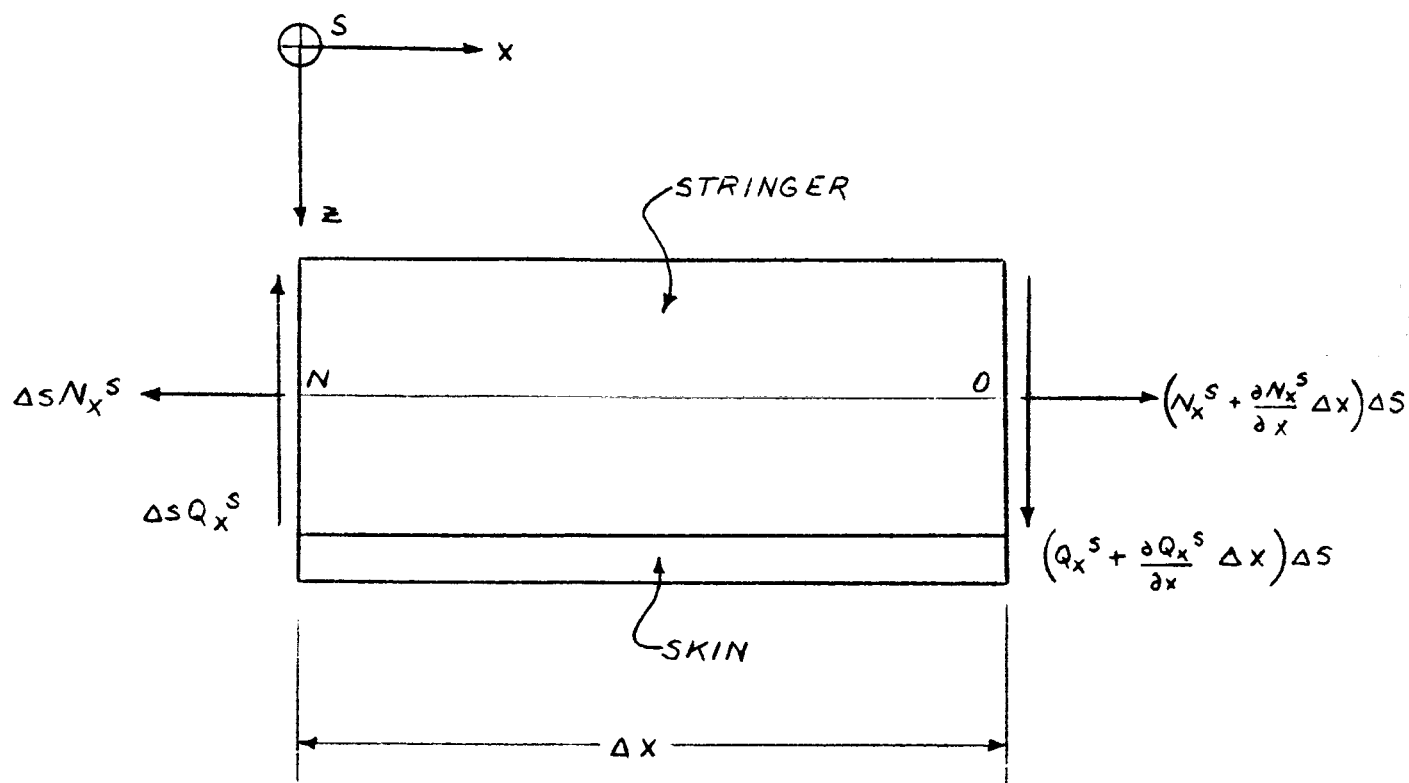


FIGURE 4. STRESS RESULTANTS DUE TO STRINGER DEFORMATION

PARAMETER	F-1	AC-2
a (in.)	60	60
Δ^f (in.)	14.3	8.27
Δ^s (in.)	8.36	8.36
L (in.)	157	157
h (in.)	.032	.032
I^s (in. ⁴)	.029	.029
A^a (in. ²)	.235	.235
I^f (in. ²)	.1414	.657
A^f (in. ⁴)	.216	.569
ν	.333	.333
E (lbs/in. ²)	10^7	10^7
$\bar{\epsilon}_s$ (in.)	.366	.366
$\bar{\epsilon}_f$ (in.)	1.120	1.120
W_s (lbs.)	570	1117
W_t (lbs.)	623	1539
K_n^s	1.422	2.91
K_n^x	1.78	1.78
K_m^s	3220	25,800
K_m^x	889	889

TABLE I. PROPERTIES OF THE F-1, AC-2 ATLAS-CENTAUR ADAPTERS

n m	2	3	4	5	6
1	91	58 49	59 59	77 82	106
2		128	100 84	100 91	118 103
3			146	130	139 125

TABLE II. FORT WORTH TEST RESULTS (BOTTOM NUMBER) AND CALCULATED FREQUENCIES (C.P.S.) FOR THE F-1 CENTAUR ADAPTER. CALCULATED FREQUENCIES ARE BASED ON AN ADAPTER WEIGHT OF 597 LBS.

m \ n					
	2	3	4	5	6
1	67	63	91	142	202
	57	53	78	121	172
2	152	104	110	148	206
	130	89	94	126	176
3	200	145	138	162	212
	171	124	118	138	182

TABLE III. CALCULATED FREQUENCIES (C.P.S.) FOR THE AC-2 CENTAUR ADAPTER.
 UPPER FREQUENCY IS BASED ON AN ADAPTER WEIGHT OF 1117 LBS.
 LOWER FREQUENCY IS BASED ON AN ADAPTER WEIGHT OF 1539 LBS.

REFERENCES

1. Fontenot, L. L. "On the Equations of Motion of Thin Pressurized Cylindrical Shells," General Dynamics/Astronautics Report ER⁴-AN-137, April 1962.
2. Reissner, E. "On Transverse Vibrations of Thin, Shallow Elastic Shells," Quarterly of Applied Mathematics, Vol. 13, pp. 169-176, 1955

The following document was also used as a reference, although not specifically referred to in the text.

1. Mitchell, R. R., "Supersonic Flutter of A Cylindrical Shell with Application to the Centaur Interstage Adapter," General Dynamics/Astronautics Report AY62-0069, January 1963.

APPENDIX A

APPROXIMATE METHOD FOR DETERMINING THE EFFECT OF PARAMETER CHANGES ON THE LOWEST FREQUENCIES OF VIBRATION OF A STIFFENED CYLINDRICAL SHELL

This section contains a detailed investigation of the frequency equation (35). The minimum frequency (with respect to circumferential mode number, n) is determined, and it is found that an approximate simple relationship between minimum frequency and shell parameters can be determined. The dependence of the other frequencies on the shell parameters is discussed in a qualitative manner. This is possible because of the observations made in the process of using (35) to compute natural frequencies for the F-1, AC-2, AC-4, AC-5 Centaur adapters and OAO aft fairing.

The frequency equation (35) for stiffened cylindrical shells is, approximately

$$a^2 k_u^2 = \frac{K_n^s K_n^x - \nu^2}{K_n^x + \phi} + \alpha a^4 (M^4 K_m^x + N^4 K_m^s) \quad (A1)$$

where

$$\phi = \frac{2N^2}{M^2} \left(\frac{K_n^x K_n^s - \nu}{1 - \nu} \right) + \frac{N^4}{M^4} K_n^s \quad (A2)$$

The quantity

$$\frac{K_n^s K_n^x - \nu^2}{K_n^x + \phi} \quad (A3)$$

will be referred to as the "stretching component of the circular frequency" and

$$\alpha a^4 (M^4 K_m^x + N^4 K_m^s) \quad (A4)$$

will be referred to as the "bending component of the circular frequency." A bar under quantities that are a function of circumferential mode numbers, n , will be used to indicate that they are evaluated at the value of n for which the circular frequency is a minimum, i.e., \underline{n} . Using this convention, the value of the lowest frequency is

$$\underline{\omega}^2 (m) \quad (A5)$$

The manner in which (A5) is written indicates $\underline{\omega}^2$ is a function of longitudinal mode number, m . That is, there will be minimum frequency for each value of m . $\underline{\omega}$ is found by setting

$$\frac{d}{dn} (a^2 k \omega^2) = 0 \quad (A6)$$

Inserting the expression for $a^2 k \omega^2$ into (A6) results in

$$\frac{K_n^x K_n^s - v^2}{K_n^x + \phi} = \alpha_a^4 \frac{N^4}{M^4} K_m^s \left\{ 1 + \frac{M^2}{N^2} \frac{1}{K_n^s} \left(\frac{K_n^x K_n^s - v}{1 - v} \right) + \frac{K_n^x - \frac{1}{K_n^s} \left(\frac{K_n^x K_n^s - v}{1 - v} \right)^2}{\frac{N^2}{M^2} \left(\frac{K_n^x K_n^s - v}{1 - v} \right) + \frac{N^4}{M^4} K_n^s} \right\} \quad (A7)$$

Equation (A7) implies that the minimum frequency occurs when the stretching term is approximately equal to the bending term. This follows from the observations that

$$\frac{N^4}{M^4} \frac{K_m^s}{K_m^x} \gg 1 \quad (A8)$$

and

$$1 + \frac{M^2}{N^2} \frac{1}{K_n^s} \left(\frac{K_n^x K_n^s - \nu}{1 - \nu} \right) + \frac{K_n^x - \left(\frac{K_n^x K_n^s - \nu}{1 - \nu} \right)^2 \left(\frac{1}{K_n^s} \right)}{\frac{N^2}{M^2} \left(\frac{K_n^x K_n^s - \nu}{1 - \nu} \right) + \frac{N^4}{M^4} K_n^s} \approx 1 \quad (A9)$$

are approximately true for most applications. The conclusion that the minimum frequency, ω , occurs when the stretching component of the circular frequency is approximately equal to the bending component of the circular frequency is verified by the calculations from the Centaur adapters. These calculations are presented graphically by Figures A-1, A-2.

An investigation of the behavior of the stretching component of the circular frequency

$$\frac{K_n^s K_n^x - \nu^2}{K_n^x + \phi} = \xi \quad (A10)$$

shows that ξ is almost constant with regard to variations on the stiffener properties (see Table A-V) and varies inversely as the cube of the circumferential mode number, n (Figure A-3). These results may be expressed mathematically as

$$\xi = C_1(m)/n^3 \quad (A11)$$

Equations (A7, A10, A11) imply that the circumferential mode number of the lowest natural frequency is given by

$$C_1(m)/n^3 = \alpha a^4 N^4 K_n^s \quad (A12)$$

and the lowest circular frequency is given by

$$a^2 k \omega^2(m) = 2 \alpha a^4 N^4 K_m^s \quad (A13)$$

When (A12) is rewritten to read

$$C_1(m) = \alpha a^7 K_m^s N^7 \quad (A14)$$

and (A14) is substituted into (A13), we have

$$a k \underline{w}^2(m) = 2 \alpha^4 K_m^s [C_1(m) / \alpha a^7 K_m^s]^{4/7} \quad (A15a)$$

i.e.

$$\underline{w}^2(m) \sim (\alpha K_m^s)^{3/7} / a^2 k \quad (A15b)$$

where \sim reads "is proportional to." Substituting the expressions for K_m^s , α

$$K_m^s = [12(1-\nu^2) I_f] [\Delta^f h^3]^{-1} \quad (A16a)$$

$$\alpha = \frac{1}{12} (h/a)^2 \quad (A16b)$$

into (A16b) results in

$$\underline{w}^2(m) \sim \left(\frac{I_f}{a^2 \Delta^f h} \right)^{3/7} (a^2 \rho')^{-1} \quad (A17)$$

where I_f = frame moment of inertia
 a = radius of cylinder
 Δ^f = distance between frames
 h = skin thickness
 ρ' = mass of reinforced cylinder/volume of skin

Equation (A17) provides a means of evaluating the change in lowest natural frequency of a stiffened cylindrical shell with changes in parameters. Equation (A17) has been applied to the AC-2, -4, -5 Centaur Adapters and the OAO aft fairing. Table A-VI shows that (A17) gives a good prediction of the change in lowest frequencies of vibration with change in parameters as compared to the complete frequency calculation via equation (A1).

The effect of parameter changes on the other natural frequencies can be expressed qualitatively by examining the form of the frequency equation (A1) and recalling the calculations that were performed for AC adapters 3, 4, 5 and the OAO aft fairing. The frequency/parameter dependence is expressed by the relation

$$\omega^2 \propto Q^P$$

where Q = parameter, such as frame inertia, etc.
 P = exponent indicating the degree of proportionality between ω^2 and Q .

Approximate values of P versus Q are given in Table A-VII. For large values of n the frequencies are proportional to n^2 (as are simple ring frequencies), i.e.

$$\omega_{n+1} / \omega_n = \left(\frac{n+1}{n} \right)^2$$

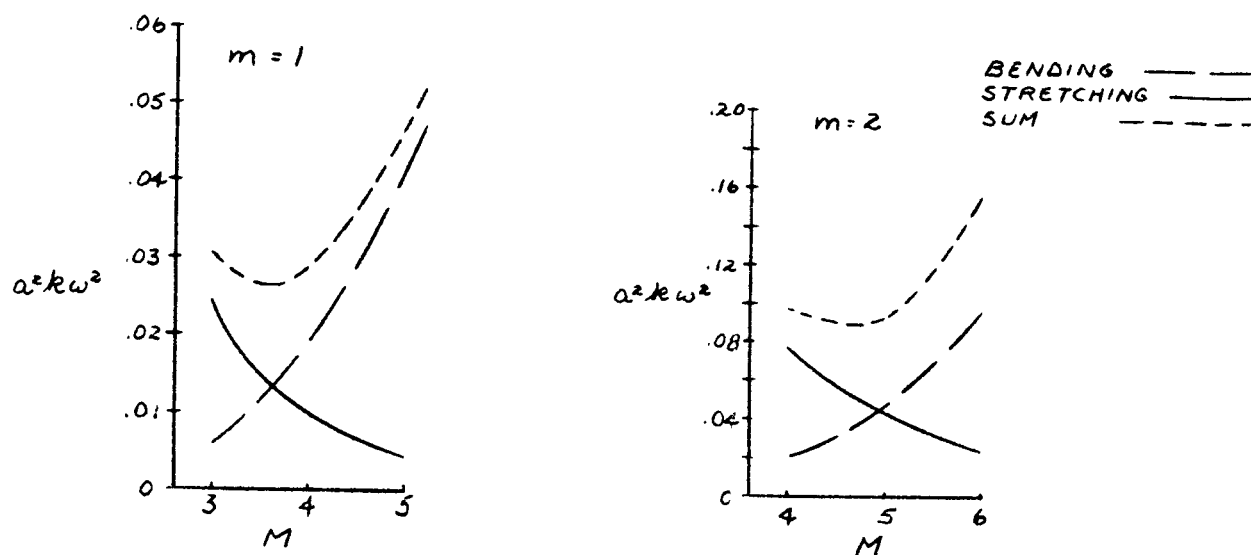


FIGURE A-1. BENDING AND STRETCHING COMPONENTS OF THE NATURAL FREQUENCIES FOR THE F-1 CENTAUR ADAPTER

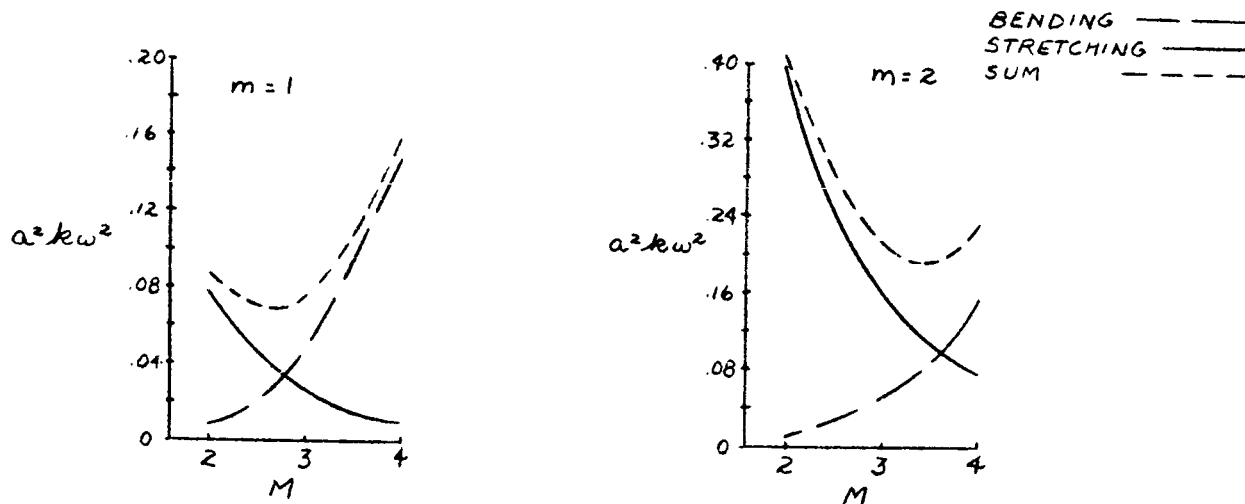


FIGURE A-2. BENDING AND STRETCHING COMPONENTS OF THE NATURAL FREQUENCIES FOR THE AC-2 CENTAUR ADAPTER

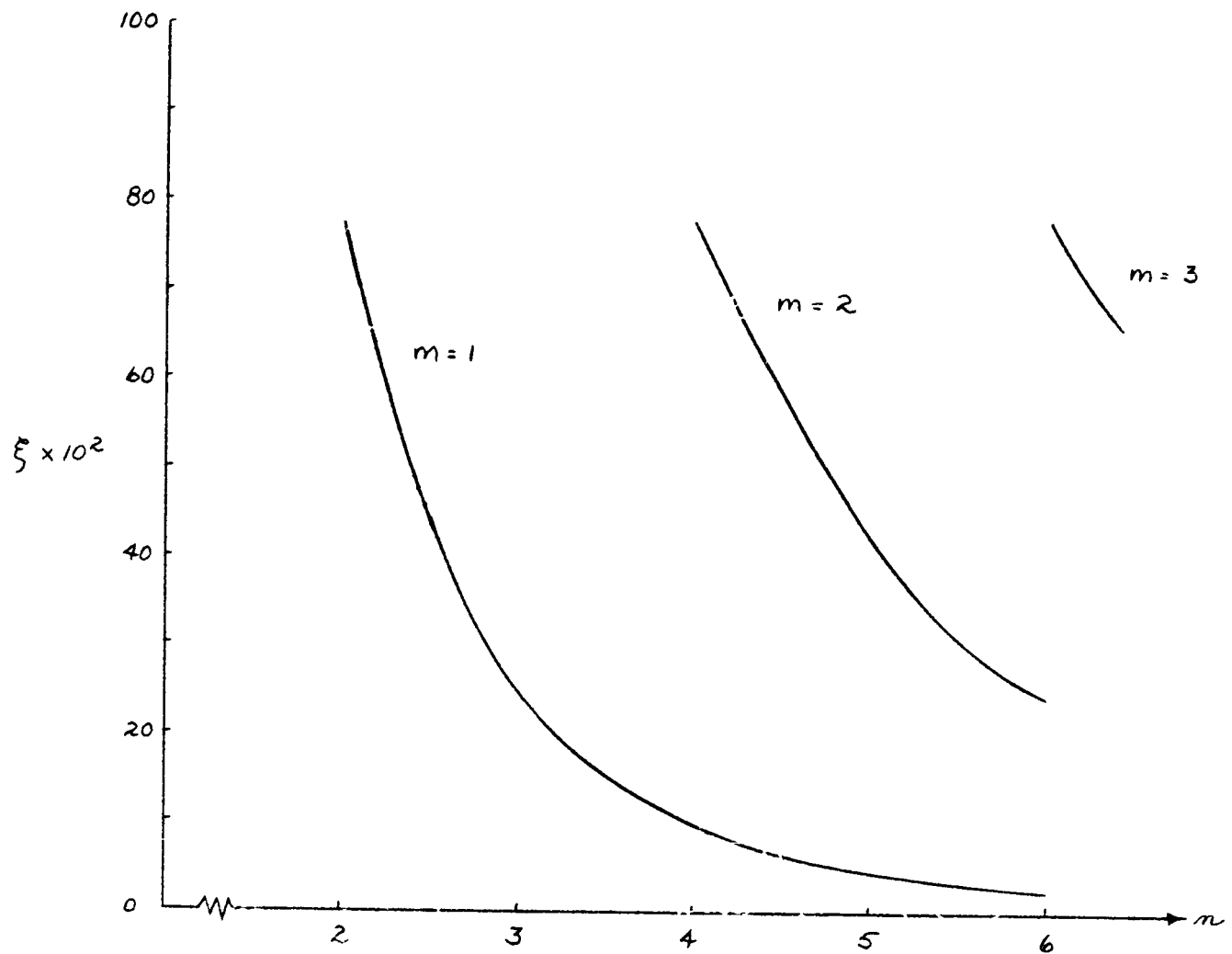


FIGURE A-3. STRETCHING COMPONENT OF THE NATURAL FREQUENCY, ξ , VERSUS CIRCUMFERENTIAL NODE NUMBER. DATA IS FROM TABLE A-V.

PARAMETER	AC-4	AC-5 (ver. 1)	AC-5 (ver. 2)	OAO
a (in.)	60	60	60	60
Δ^f (in.)	26.2	14.3	26.2	14.50
Δ^s (in.)	8.36	2.09	2.09	6.30
L (in.)	157	157	157	157
h (in.)	.04	.04	.04	.032
I^s (in. ⁴)	.0326	.00589	.00589	.0168
A^s (in. ²)	.260	.0769	.0769	.159
I^f (in. ⁴)	.651	1.51	2.41	.146
A^f (in. ²)	.569	.589	.970	.185
ν	.333	.333	.333	.333
E (lbs/in. ²)	10^7	10^7	10^7	10^7
$\bar{\epsilon}_s$ (in.)	.366	.500	.500	.342
$\bar{\epsilon}_f$ (in.)	1.12	2.01	2.01	1.18
W_s (lbs.)	1100	1150	1000	
W_t (lbs.)	1400	1450	1300	678
K_n^s	1.483	1.917	1.824	1.354
K_n^x	1.692	1.819	1.819	1.701
K_m^s	4191	17,600	15,350	3,290
K_m^x	652	470	470	865

TABLE A-I. PROPERTIES OF THE ATLAS-CENTAUR ADAPTERS AC-4, AC-5
(VERSIONS 1, 2) AND THE OAO AFT FAIRING

$\begin{matrix} n \\ m \end{matrix}$	2	3	4	5	6
1	64	43	51	72	102
2	132	91	76	84	107
3	177	131	108	104	119

TABLE A-II. CALCULATED FREQUENCIES (C.P.S.) FOR THE PROPOSED VERSION OF THE AC-4 CENTAUR ADAPTER. CALCULATIONS ARE BASED ON AN ADAPTER WEIGHT OF 1400 LBS.

$\begin{matrix} n \\ m \end{matrix}$	2	3	4	5	6
1	60	56	84	129	184
	62	56	83	125	181
2	122	92	99	135	188
	128	109	99	132	184
3	168	130	122	147	193
	173	134	125	146	190

TABLE A-III. CALCULATED FREQUENCIES (C.P.S.) FOR TWO PROPOSED VERSIONS OF THE AC-5 CENTAUR ADAPTER. THE TOP FREQUENCY IS FOR VERSION 1 (10 FRAMES, $w = 1450$ LBS.) THE BOTTOM FREQUENCY IS FOR VERSION 2 (5 FRAMES, $w = 1400$ LBS.)

$\begin{matrix} n \\ m \end{matrix}$	2	3	4	5	6
1	82	53	53	69	94
2	168	114	89	88	106
3	221	164	128	114	121

TABLE A-IV. CALCULATED FREQUENCIES (C.P.S.) FOR THE OAO AFT FAIRING. CALCULATIONS ARE BASED ON A FAIRING WEIGHT OF 678 LBS.

m	n	AC-1	AC-3	AC-4	AC-5, #1	AC-5, #2	OAO
		$K_n^s = 1.422$ $K_n^x = 1.78$	$K_n^s = 2.91$ $K_n^x = 1.78$	$K_n^s = 1.483$ $K_n^x = 1.692$	$K_n^s = 1.917$ $K_n^x = 1.819$	$K_n^s = 1.824$ $K_n^x = 1.819$	$K_n^s = 1.354$ $K_n^x = 1.701$
1	2	.0777	.0781	.0770	.0779	.0786	.0773
1	3	.0245	.0250	.0240	.0248	.0246	.0242
1	4	.0100	.0101	.0098	.0102	.0100	.0095
1	5	.0045	.0046	.0043	.0046	.0046	.0044
1	6	.0023	.0023	.0020	.0023	.0024	.0022
2	2			.3460	.3460	.3600	.3400
2	3	.1557	.1610	.1560	.1595	.1580	.1543
2	4	.0776	.0781	.0770	.0780	.0788	.0773
2	5	.0435	.0424	.0416	.0428	.0429	.0418
2	6	.0245	.0245	.0240	.0248	.0248	.0242
3	2			.6270	.6900	.6670	.5980
3	3			.3460	.3630	.3600	.3400
3	4	.2000	.2100	.2010	.2060	.2040	.1984
3	5	.1230	.1250	.1220	.1250	.1250	.1212
3	6	.0779	.0785	.0770	.0780	.0786	.0773

TABLE A-V. VARIATION OF STRETCHING COMPONENT, ξ , WITH FRAME AND STRINGER STIFFNESSES K_n^f , K_n^s AND MODE NUMBERS m , n .

	ATLAS-CENTAUR ADAPTERS				OAO AFT FAIRING
	AC-3	AC-4	AC-5 (ver. 1)	AC-5 (ver. 2)	
$\frac{\omega}{\omega_r}$ (from A1)	.98	.84	1.09	.92	.92
$\frac{\omega}{\omega_r}$ (from A18)	.93	.80	1.02	1.01	.91
% difference	+5%	+5%	+4%	-9%	+1%

TABLE A-VI. COMPARISON OF FREQUENCY RATIO $(\frac{\omega}{\omega_r})_{m=1}$, PREDICTED BY A COMPLETE CALCULATION (A1) VERSUS FREQUENCY RATIO PREDICTED BY THE APPROXIMATE FORMULA (A18). THE REFERENCE CONFIGURATION IS THE F-1 CENTAUR ADAPTER.

Q	$P(n < \underline{n})$	$P(n > \underline{n})$
I_f	$-\infty$	+1
I_s	$-\infty$	$-\infty$
h	-1	-1

TABLE A-VII. VARIATION OF THE SQUARE OF THE FREQUENCY (ω^2) WITH PARAMETER VARIATION. $\omega^2 \propto Q^p$, \underline{n} = CIRCUMFERENTIAL MODE NUMBER AT MINIMUM VALUE OF ω .